Folds in Haskell

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Folds!

- A list xs can be built by applying the (:) and [] operators to a sequence of values:

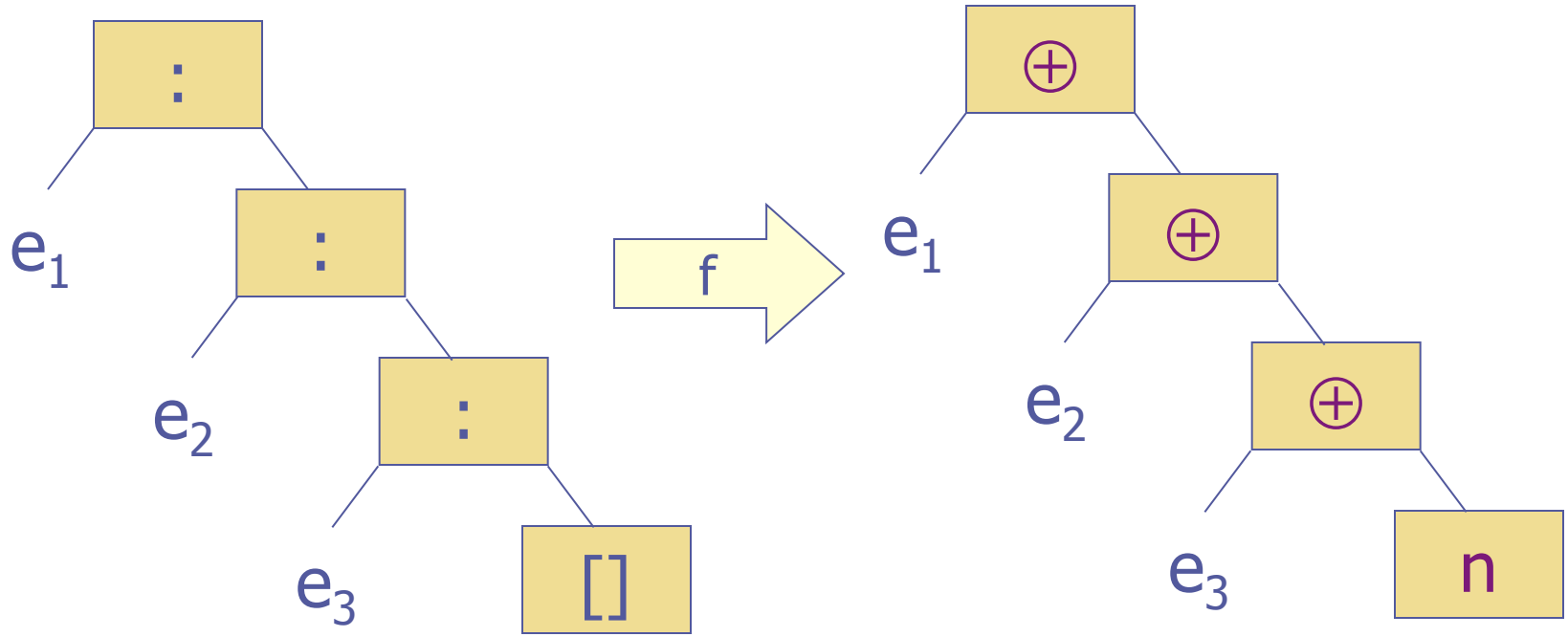
$$xs = x_1 : x_2 : x_3 : x_4 : \ldots : x_k : []$$

- Suppose that we are able to replace every use of (:) with a binary operator ($\oplus$), and the final [] with a value n:

$$xs = x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus \ldots \oplus x_k \oplus n$$

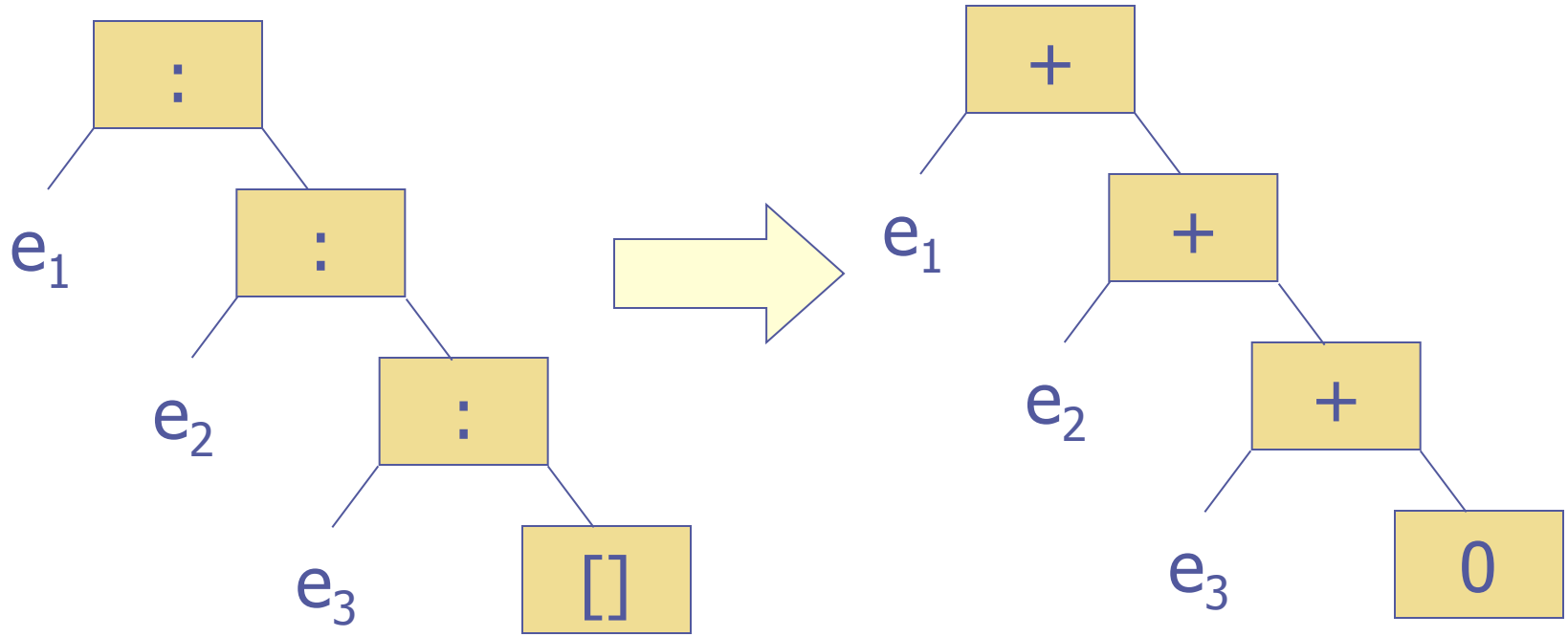
- The resulting value is called fold ($\oplus$) n xs
- Many useful functions on lists can be described in this way.

Graphically:



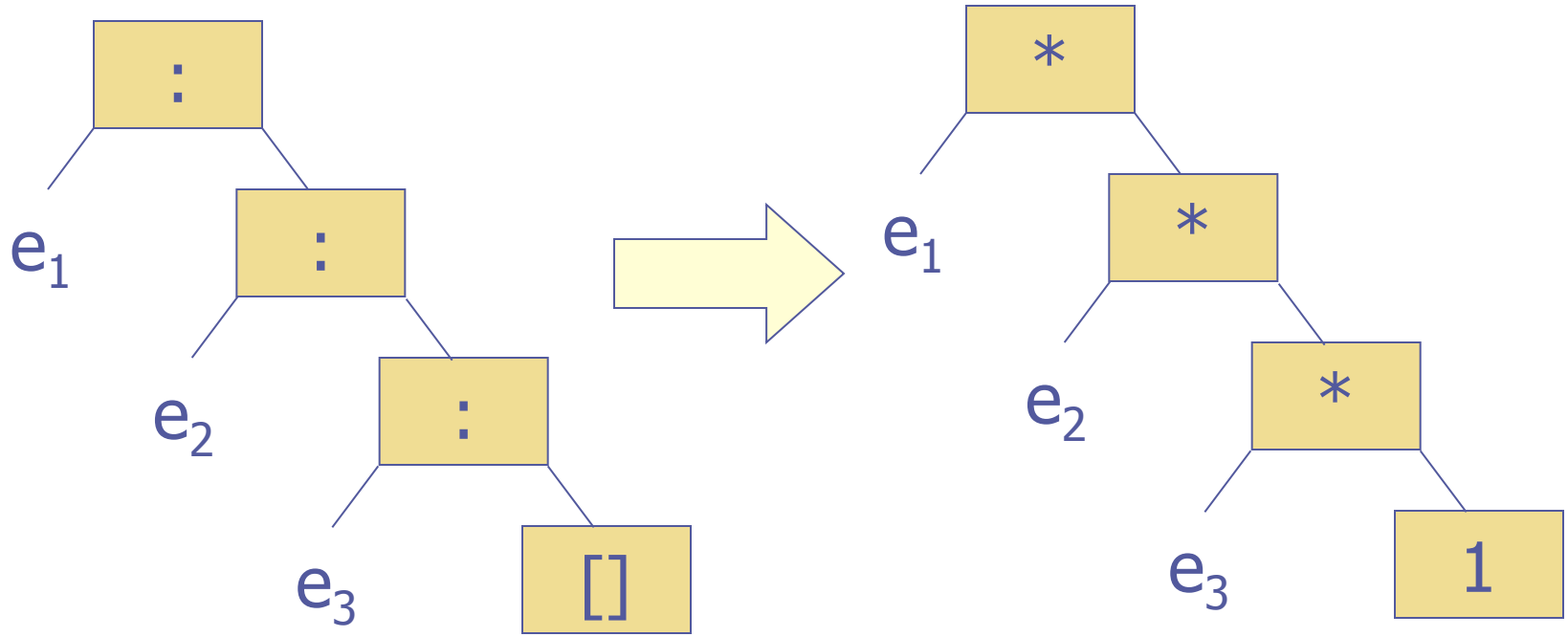
f = foldr (⊕) n

Example: sum



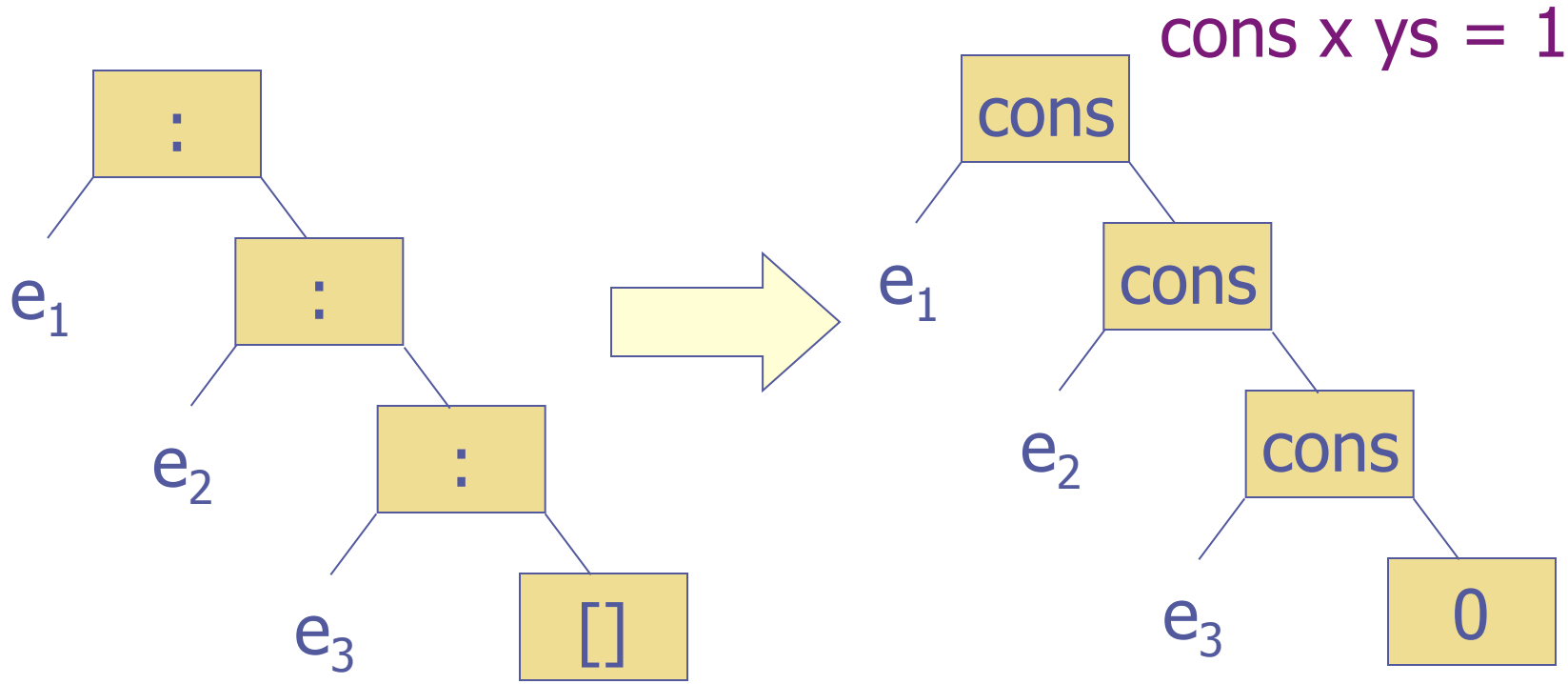
sum = foldr (+) 0

Example: product



product = foldr (*) 1

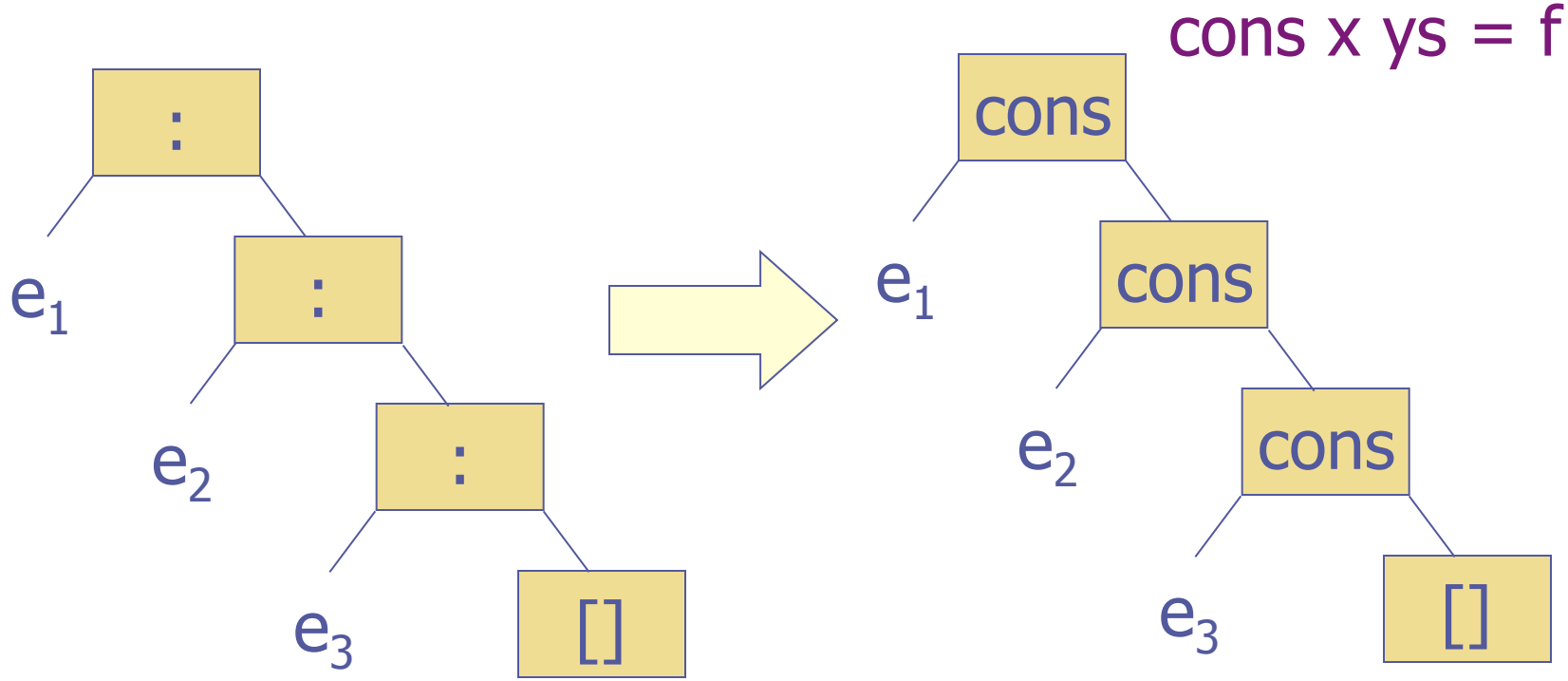
Example: length



length = foldr (\x ys -> 1 + ys) 0

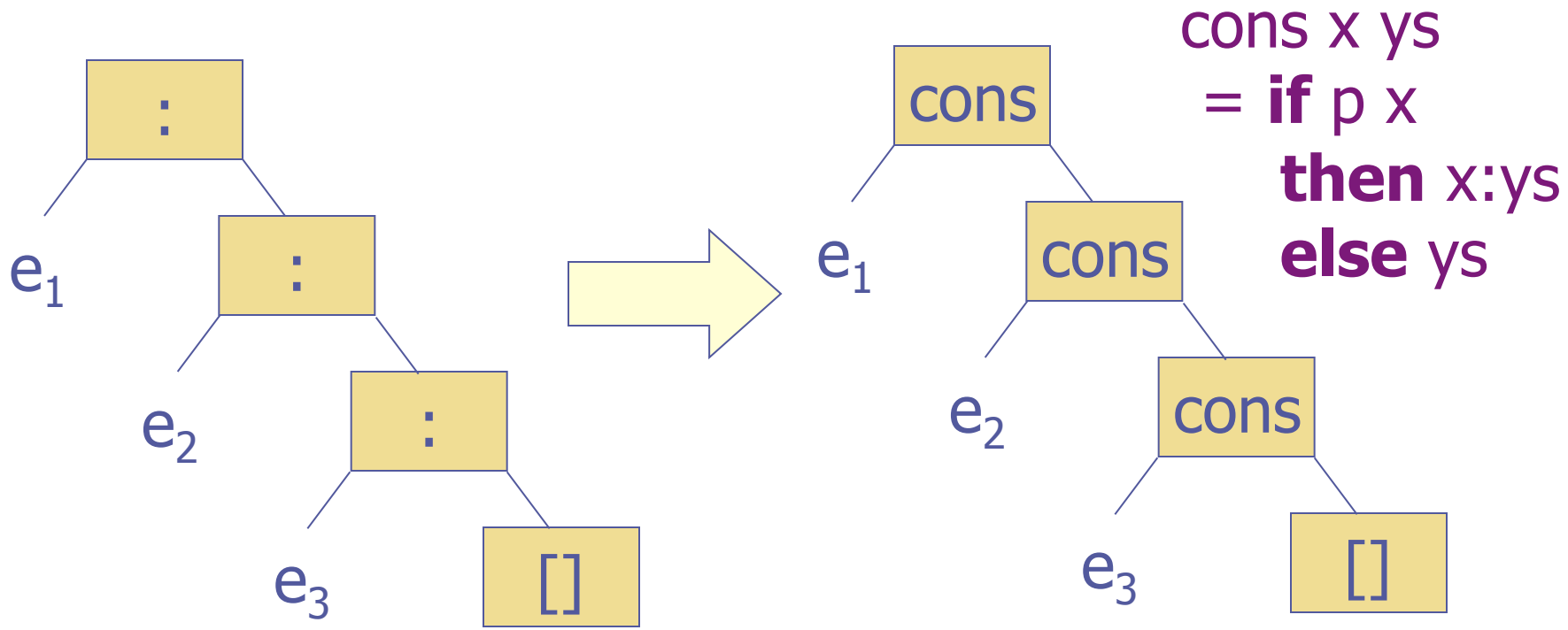
Example: map

cons x ys = f x:ys



map f = foldr (\x ys -> f x : ys) []

Example: filter



filter p = foldr (\x ys -> **if** p x **then** x:ys **else** ys) []

Formal Definition:

```
foldr                    :: (a->b->b) -> b -> [a] -> b
foldr cons nil []         = nil
foldr cons nil (x:xs)     = cons x (foldr cons nil xs)
```

Applications:

```
sum             = foldr (+) 0
product         = foldr (*) 1
length          = foldr (\x ys -> 1 + ys) 0
map f           = foldr (\x ys -> f x : ys) []
filter p        = foldr c []
  where c x ys = if p x then x:ys else ys
xs ++ ys        = foldr (:) ys xs
concat          = foldr (++) []
and             = foldr (&&) True
or              = foldr (||)   False
```

Patterns of Computation:

- foldr captures a common pattern of computations over lists
- As such, it's a very useful function in practice to include in the Prelude
- Even from a theoretical perspective, it's very useful because it makes a deep connection between functions that might otherwise seem very different …
- From the perspective of lawful programming, one law about foldr can be used to reason about many other functions

A law about foldr:

- If $(\oplus)$ is an associative operator with unit n, then

  foldr $(\oplus)$ n xs $\oplus$ foldr $(\oplus)$ n ys
  = foldr $(\oplus)$ n (xs ++ ys)

- $(x_1 \oplus \ldots \oplus x_k \oplus n) \oplus (y_1 \oplus \ldots \oplus y_j \oplus n)$
  $= (x_1 \oplus \ldots \oplus x_k \oplus y_1 \oplus \ldots \oplus y_j \oplus n)$

- All of the following laws are special cases:

```
sum xs      +    sum ys      = sum (xs ++ ys)
product xs  *    product ys  = product (xs ++ ys)
concat xss  ++   concat yss  = concat (xss ++ yss)
and xs      &&   and ys      = and (xs ++ ys)
or xs       ||   or  ys      = or (xs ++ ys)
```

foldl:

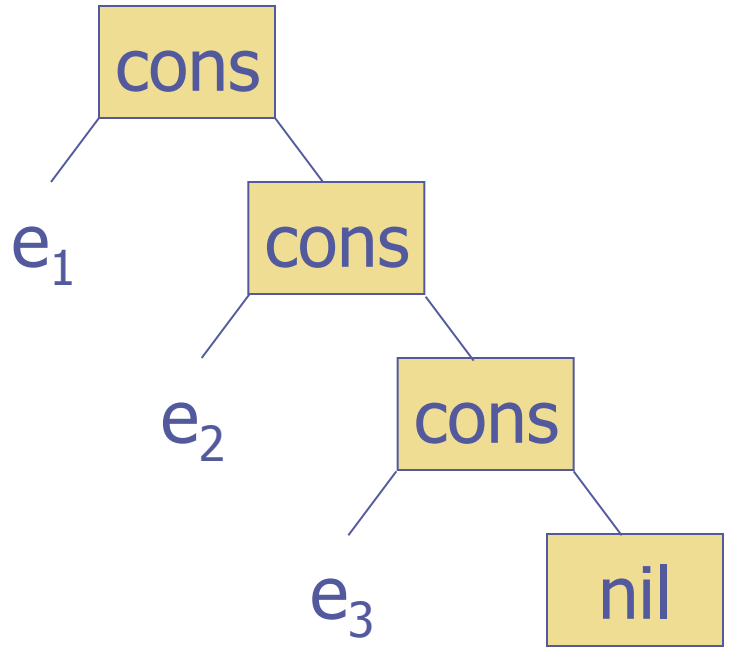
- There is a companion function to foldr called foldl:

```
foldl             :: (b -> a -> b) -> b -> [a] -> b
foldl s n []      = n
foldl s n (x:xs)  = foldl s (s n x) xs
```

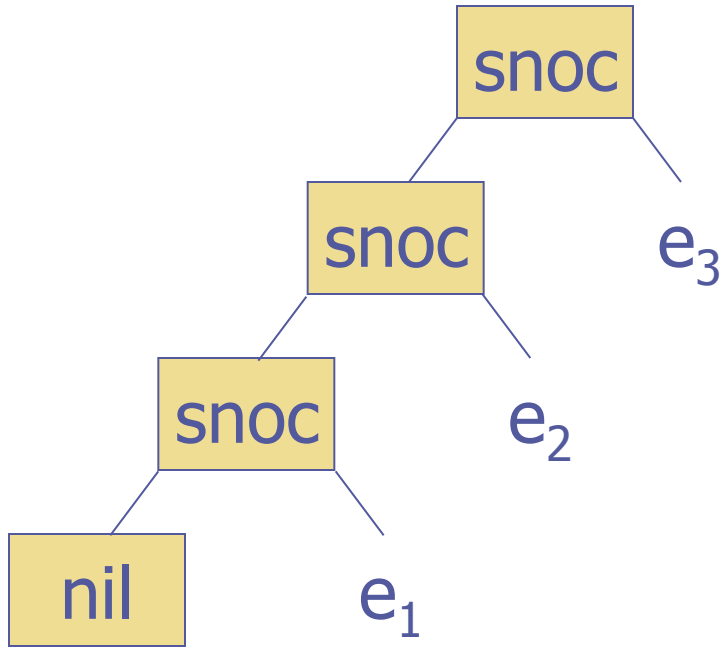
- For example:

$$\begin{aligned} &\texttt{foldl s n } [e_1, e_2, e_3] \\ &\quad = \texttt{s (s (s n } e_1\texttt{) } e_2\texttt{) } e_3 \\ &\quad = \texttt{((n `s` } e_1\texttt{) `s` } e_2\texttt{) `s` } e_3 \end{aligned}$$

foldr vs foldl:



foldr



foldl

Uses for foldl:

- Many of the functions defined using foldr can be defined using foldl:

```
sum     = foldl (+) 0
product = foldl (*) 1
```

- There are also some functions that are more easily defined using foldl:

```
reverse = foldl (\ys x -> x:ys) []
```

- When should you use foldr and when should you use foldl? When should you use explicit recursion instead?

foldr1 and foldl1:

- Variants of foldr and foldl that work on non-empty lists:

```
foldr1              :: (a -> a -> a) -> [a] -> a
foldr1 f [x]        = x
foldr1 f (x:xs)     = f x (foldr1 f xs)

foldl1              :: (a -> a -> a) -> [a] -> a
foldl1 f (x:xs)     = foldl f x xs
```

- Notice:
  - No case for empty list
  - No argument to replace empty list
  - Less general type (only one type variable)

Uses of foldl1, foldr1:

From the prelude:

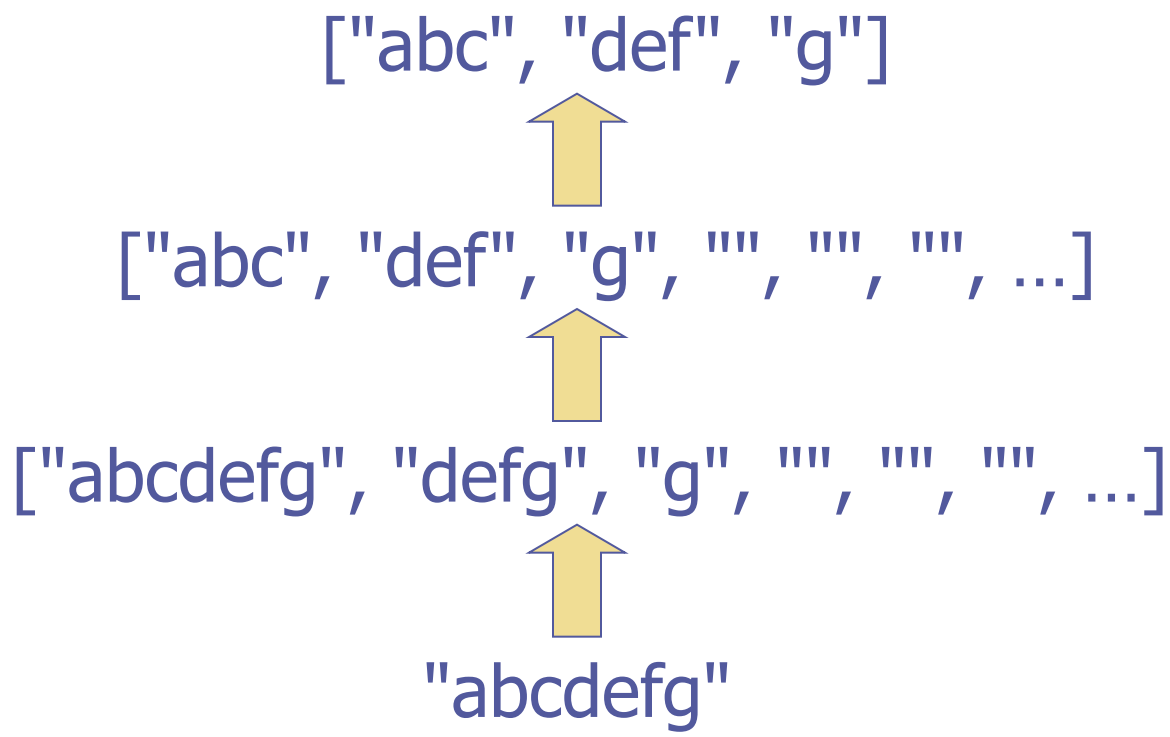
```
minimum = foldl1 min
maximum = foldl1 max
```

Not in the prelude:

```
commaSep = foldr1 (\s t -> s ++ ", " ++ t)
```

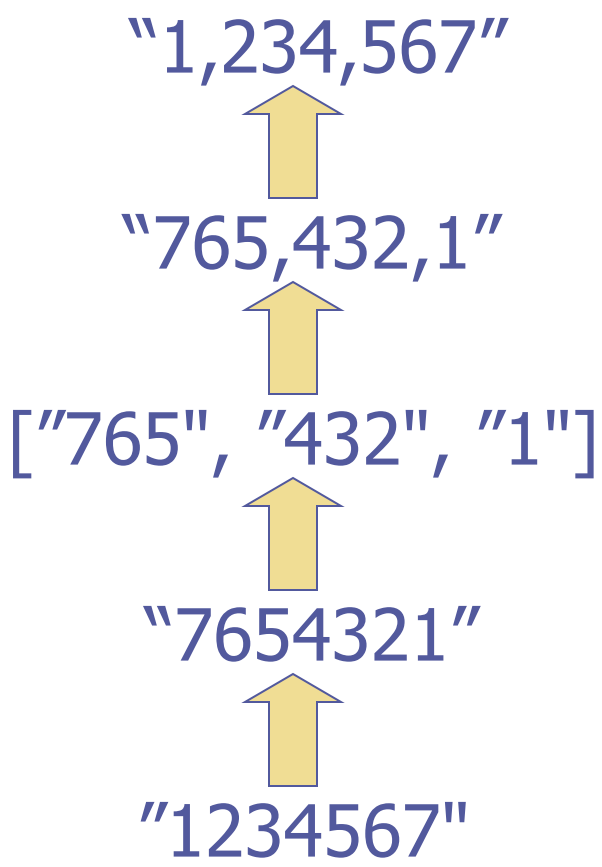
Example: Grouping

```
group n      =   takeWhile (not.null)
                 .   map (take n)
                 .   iterate (drop n)
```



Example: Adding Commas

```
group n   = reverse
          . foldr1 (\xs ys -> xs++","++ys)
          . group 3
          . reverse
```



Example: transpose

```
transpose                 :: [[a]] -> [[a]]
transpose []              = []
transpose ([] : xss)      = transpose xss
transpose ((x:xs) : xss)
                  = (x : [h | (h:t) <- xss])
                    : transpose (xs : [ t | (h:t) <- xss])
```

Example:

transpose [[1,2,3],[4,5,6]] = [[1,4],[2,5],[3,6]]

Example: say

```
Say> putStr (say "hello")


H    H  EEEEE  L       L         OOO
H    H  E      L       L       O   O
HHHHH  EEEEE  L       L       O   O
H    H  E      L       L       O   O
H    H  EEEEE  LLLLL   LLLLL     OOO


Say>
```

… continued:

```
say   = ('\n':)
      . unlines
      . map (foldr1 (\xs ys->xs++"  "++ys))
      . transpose
      . map picChar
 where
  picChar 'A' = [  "  A  ",
                   " A A ",
                   "AAAAA",
                   "A   A",
                   "A   A" ]
 etc…
```

Composition and Reuse:

```
Say> (putStr . concat . map say . lines . say) "A"
                A
               A A
              AAAAA
              A   A
              A   A

         A             A
        A A           A A
       AAAAA         AAAAA
       A   A         A   A
       A   A         A   A

  A      A      A      A      A
 A A    A A    A A    A A    A A
AAAAA  AAAAA  AAAAA  AAAAA  AAAAA
A   A  A   A  A   A  A   A  A   A
A   A  A   A  A   A  A   A  A   A

  A                           A
 A A                         A A
AAAAA                       AAAAA
A   A                       A   A
A   A                       A   A

  A                           A
 A A                         A A
AAAAA                       AAAAA
A   A                       A   A
A   A                       A   A

Say>
```

Summary:

- Folds on lists have many uses
- Folds capture a common pattern of computation on list values
- In fact, there are similar notions of fold functions on many other algebraic datatypes …)